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INVESTIGATION OF THE METHOD OF DETERMINING THE RELATION OF STATICAL HYSTERESIS AND FLEXURAL STRESS BY MEASUREMENT OF THE DECREMENT OF A FREELY VIBRATING U BAR

By G. H. Keulegan

ABSTRACT

This investigation was undertaken to determine whether measurements of the decrement of a vibrating U bar could be used to measure with sufficient accuracy the statical hysteresis of the material. The statical hysteresis of a U bar of Armco iron was first measured under cyclic static loading, and then the decrement of the vibrations of the same bar were measured. The results of the experiments showed that within the limits of accuracy of the approximate theory used, both methods gave equivalent results. For stresses above a certain small threshold value, it was found that in Armco iron the energy lost by statical hysteresis varied approximately as the cube of the amplitude of the maximum stress.

CONTENTS

I. INTRODUCTION

This investigation is part of a general investigation of statical hysteresis in metals. Three reports have been previously published,[1 2 3] in the first and second of which it is shown, in common with the results of other workers in this field, that the statical hysteresis in flexure is, to the first order, a function of the stress alone. This report is a continuation of the first report,[4] extending the experimental data and involving the investigation of a different and more convenient method of experimentation.

[1] G. H. Keulegan, Statical Hysteresis in the Flexure of Bars, B. S. Tech. Paper No. 332, 1926.
[2] G. H. Keulegan, Statical Hysteresis in Cycles of Equal Load Range, B. S. Tech. Paper No. 365, 1928.
[3] G. H. Keulegan, On the Vibration of U Bars, B. S. Jour. Research, vol. 6, pp. 553-592, 1931.
[4] See footnote 1.

Methods used by other investigators of hysteresis phenomena include measurement of (*a*) static deformation of rods in uniform flexure,[5] (*b*) static deformation of strips in elongation [6]; (*c*) the rise of temperature of rods in alternate, axial stress cycles [7]; (*d*) the amplitude at resonance during the forced oscillations of tubes in flexure [8]; (*e*) the distortion of loaded rotating rods [9]; (*f*) the decrement of the longitudinal vibrations of a rod [10]; and (*g*) the torque of a Rayleigh disk in the proximity of a longitudinally vibrating rod [11]; the decay of the lateral oscillations of clamped and loaded strips.[12] It is planned to consider the results obtained by these methods in a future paper.

The term statical hysteresis refers, as in the earlier reports,[13] to the component of elastic lag which is independent of rate of loading of the elastic body during the load cycle. The elastic lag is the difference in deflection of an elastic body at any load in a load cycle in which the load is first increased to higher values and then decreased to the original value again. The component of the elastic lag dependent on the rate of loading is due to elastic after working and is called hereditary hysteresis.

The direct method of measuring elastic lag involves applying a series of loads at a given rate to the body under investigation and measuring the corresponding deflections. This method is difficult to use in an extensive investigation owing to the small values of the elastic lag and the large number of independent readings which must be taken. For this reason the present investigation was undertaken to determine if measurements of the decrements of the amplitude of a freely vibrating U bar would give comparable results for materials subjected to flexural stresses. As the loss in energy is all that can be determined from the decrement, the elastic hysteresis must be expressed in terms of the loss in energy per cycle, but this is no disadvantage since the relation of elastic hysteresis to load (or deflection) is well known and its magnitude can be calculated if the energy loss per cycle is known.

In general, the damping of a vibrating body is brought about by the combined effects of elastic afterworking and statical hysteresis. If the effect of elastic afterworking is negligible in comparison with that of statical hysteresis, the vexing question of the time element is eliminated from consideration, and the hysteresis calculated from the decrement of a vibrating body should be directly comparable with that calculated from slowly alternating static deformations. Armco iron appears to be a material in which the elastic afterworking is negligible. Consequently, the experimental data were obtained on a U bar of this material.

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[5] Sayre, M. F., and Hoadley, A. Stress Distribution and Hysteresis Losses in Springs, Applied Mechanics (A. S. M. E. Trans.), vol. 51, pp. 287–303, 1929.

[6] Sayre, M. F. Elastic and Inelastic Behaviour in Spring Materials, Applied Mechanics (A. S. M. E. Trans.), vol. 52, pp. 105–111, 1930.

[7] B. Hopkinson and G. T. Williams, London, Proc. R. Soc., vol. 87, p. 502; 1912.

[8] R. H. Canfield, Internal Friction in Metals, Phys. Rev., vol. 32, pp. 521–530, 1928.

[9] A. L. Kimball and D. E. Lovell, Internal Friction in Solids, Phys. Rev., vol. 30, pp. 948–959, 1927.

[10] E. Voigt. Eine neue Methode zur Bestimmung der inneren Arbeitsaufnahmefähigkeit von Werksoffen bei dynamischer Beanspruchung, Zeit. f. Tech. Phys., vol. 9, pp. 321–337, 1928.

[11] S. L. Quimby, On the Experimental Determination of the Viscosity of Vibrating Solids, Phys. Rev., vol. 25, pp. 558–569, 1925.

[12] K. Honda and S. Konno, On the Determination of the Coefficients of Normal Viscosity of Metals. Phil. Mag., vol. 42, pp. 115–123, 1921.

[13] See footnotes 1 and 2, p. 635.

Two sets of independent data were obtained on the U bar, (*a*) when stressed flexurally by means of a series of static loads and (*b*) when vibrating freely. The results, in the light of the theory adopted, are shown to be equivalent, thus indicating that the decrement of the amplitude of vibration is sufficient to determine the hysteresis of materials of the type where elastic afterworking is negligible in comparison with statical hysteresis.

II. THEORY OF THE ENERGY LOSS IN A STRAIGHT BAR DUE TO STATICAL HYSTERESIS

1. DURING ALTERNATING FLEXURE PRODUCED BY A LOAD CYCLE

Consider a straight rectangular bar of uniform thickness $2a$, width b, and length s_1 (see fig. 1) clamped at one end and a load L

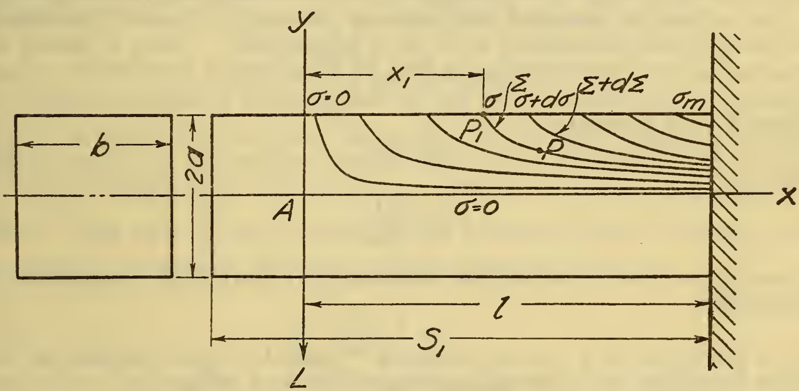

FIGURE 1.—*Lines of equal stress amplitude σ in the upper section of the clamped-free bar in the plane of bending*

applied at some point A. Let l denote the distance of this point from the fixed end. (Hereafter, l will be called the effective length of the bar in flexure.) The alternating flexure of the bar is produced by varying the load L between the limits $+L_m$ and $-L_m$ in a cyclic manner. (Hereafter L_m will be called the amplitude of the alternating load in the cycle.) Construct a set of coordinate axis with the origin at A, the x axis in the neutral plane of the bar and the y axis in the plane of bending. During a cycle of alternating flexure the material at a point P (x, y) (see fig. 1) undergoes a cycle of alternating stress of amplitude σ, which, in terms of the amplitude of the alternating load, is

$$\sigma = xy\frac{L_m}{I} \tag{1}$$

where I is the moment of inertia of the section of the bar at P. For a rectangular bar

$$I = \frac{2}{3}ba^3 \tag{2}$$

During each cycle of alternating stress statical hysteresis causes a loss of a certain amount of elastic energy. This loss in elastic energy

in a differential of volume, dv, represented by dH, is assumed to be proportional to the volume and a function of the stress amplitude σ at any point P in the bar; that is

$$dH = f\ (\sigma)\ dv \tag{3}$$

The form of this function is to be determined by experiment.

The total loss of energy, H, in the whole bar during a cycle of alternating flexure is

$$H = 2\int f\ (\sigma)\ dv \tag{4}$$

where the integration is made through the upper half of the bar. The next step in the calculation of H is to determine dv as a function of σ and the constants of the bar.

Equation (1) states that during a cycle of alternating flexure, points of the bar in the plane of bending (xy plane, fig. 1) and possessing the same stress amplitude σ lie on a hyperbola. Thus, a family of hyperbolae each representing a line of equal stress amplitude σ may be drawn in a section of the bar in the plane of bending as shown in Figure 1. If we write

$$\Sigma = xy \tag{5}$$

where

$$\Sigma = \frac{\sigma I}{L_m} \tag{6}$$

Σ can be used as a parameter defining any particular hyperbola of the family.

Let the hyperbola passing through P meet the upper surface of the bar at the point P_1. Let the abscissa of P_1 be x_1, then

$$\Sigma = ax_1 \tag{7}$$

Now the area F of the bar below the hyperbola with parameter Σ is

$$F = ax_1 + \int_{x_1}^{l} ydx = \Sigma + \int_{\frac{\Sigma}{a}}^{l} \frac{\Sigma}{x} dx = \Sigma + \Sigma \log \frac{al}{\Sigma} \tag{8}$$

Hence, if we differentiate F, with respect to Σ, we obtain the expression

$$dF = \left(\log \frac{al}{\Sigma}\right) d\Sigma \tag{9}$$

which is the elementary area lying between the two consecutive hyperbolae with parameters Σ and $\Sigma + d\Sigma$. All of the points in this elementary area have the same stress amplitude σ. Since the bar is of uniform width b, the volume of the elementary portion of the bar, having the same value of σ, is

$$dv = b\left(\log \frac{al}{\Sigma}\right) d\Sigma \tag{10}$$

Next, let σ_m be the maximum stress amplitude σ in the bar during a cycle of alternating flexure in which the load amplitude is L_m. This maximum stress amplitude occurs at the point $x = l$, $y = a$; that is,

at the upper and the lower surfaces of the bar in the neighborhood of the clamp. Then equation (1) becomes

$$\sigma_m = \frac{alL_m}{I} \tag{11}$$

Substituting from equations (6), (7), and (11), equation (10) becomes

$$dv = \frac{alb}{\sigma_m} \log \frac{\sigma_m}{\sigma} d\sigma \tag{12}$$

Letting V be one-half of the volume of the bar, or alb,

$$dv = \frac{V}{\sigma_m} \log \frac{\sigma_m}{\sigma} d\sigma \tag{13}$$

Substituting the value of dv from equation (13) in equation (4) we obtain the loss in energy in terms of the stress amplitude

$$H = \frac{2V}{\sigma_m} \int_0^{\sigma_m} \log \frac{\sigma_m}{\sigma} f(\sigma)\, d\sigma \tag{14}$$

It is convenient to express the results in terms of the ratio of the energy loss H to the entire energy of elastic deformation W of the bar when subjected to the maximum load L_m of the alternating load cycle. This ratio, $\frac{H}{W}$ will be called the fractional loss of energy due to statical hysteresis. In terms of the stress amplitude

$$W = \frac{1}{E} \int \sigma^2\, dv$$

where the integration is taken over the upper half of the bar. E is Young's modulus of elasticity of the bar material. Introducing the value of dv from equation (13),

$$W = \frac{V}{E\sigma_m} \int_0^{\sigma_m} \left(\log \frac{\sigma_m}{\sigma}\right) \sigma^2 d\sigma$$

and, after integration,

$$W = \frac{V}{9} \frac{\sigma_m^2}{E} \tag{15}$$

The fractional loss of energy is then:

$$\frac{H}{W} = \frac{18E}{\sigma_m^3} \int_0^{\sigma_m} \log \frac{\sigma_m}{\sigma} f(\sigma)\, d\sigma \tag{16}$$

This is the basic equation for the study of statical hysteresis in the alternating flexure of rectangular bars having one end clamped. Here, as previously stated, σ_m is the maximum stress amplitude in the bar corresponding to the load L_m.

It has been found by preliminary experiments that $f(\sigma)$ is given with sufficient approximation by the relations

$$\begin{aligned} f(\sigma) &= 0 \qquad \sigma \leqq \sigma_0 \\ f(\sigma) &= \left(1 - \frac{\sigma_0}{\sigma}\right) \sum_{r=2}^{r=n} \beta_r\, \sigma^r, \quad \sigma \geqq \sigma_0 \end{aligned} \tag{17}$$

where the quantities β_r are undetermined constant coefficients and σ_0 is a small stress, which in a particular case may be zero. This gives

$$\frac{H}{W}=\frac{18E}{\sigma_m^{\;3}}\sum_{r=2}^{r=n}\beta_r\left[-\left(\log\frac{\sigma_0}{\sigma_m}\right)\frac{\sigma_0^{\;r+1}}{r\;(r=1)}\right.$$
$$\left.+\frac{\sigma_m^{\;r+1}}{(r+1)^2}-\frac{\sigma_0\,\sigma_m^{\;r}}{r^2}+\frac{(2\,r+1)\,\sigma_0^{\;r+1}}{r^2(r+1)^2}\right] \tag{18}$$

When σ_m is large in comparison with σ_0, we can neglect small quantities and (18) becomes

$$\frac{H}{W}=\frac{18E}{\sigma_m^{\;3}}\sum_{r=2}^{r=n}\beta_r\left[\frac{\sigma_m^{\;r+1}}{(r+1)^2}-\frac{\sigma_0\,\sigma_m^{\;r}}{r^2}\right] \tag{19}$$

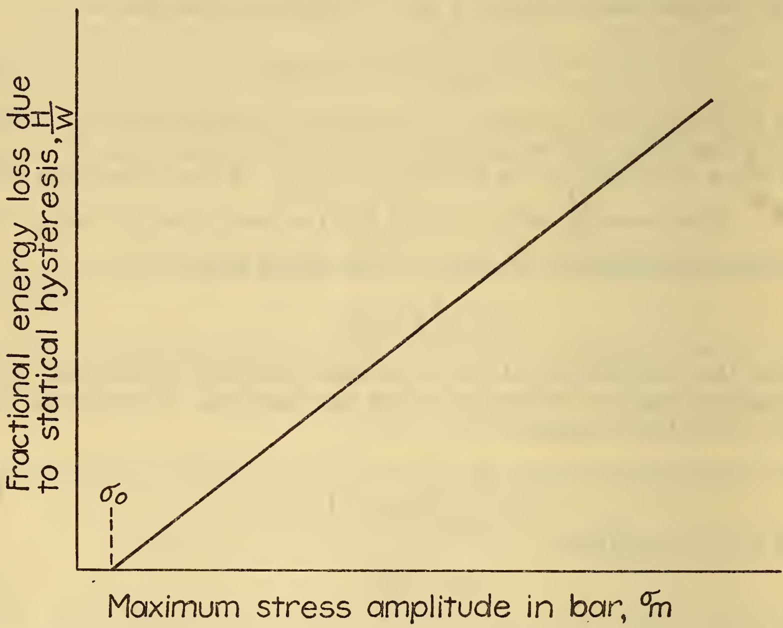

FIGURE 2.—*A typical curve showing the linear dependence of the fractional energy loss due to statistical hysteresis on the difference in the stress amplitudes* $\sigma-\sigma_0$

In practice, equation (19) is sufficiently accurate since $\frac{H}{W}$ can be accurately determined only for large values of σ_m (that is, of L_m). To determine the constants β_r in equation (17), express the experimentally determined values of $\frac{H}{W}$ as a power series in σ_m, and substitute in equation (19). Then equate the coefficients of like powers of σ_m, after experimentally obtaining the value of σ_0 as the maximum value of σ_m for which $\frac{H}{W}$ is practically zero. There will be some indefiniteness in the determination of σ_0, but the errors arising from this source are not serious.

Plotting the experimental results shows that the relation between $\frac{H}{W}$ and σ_m is very closely linear as illustrated in Figure 2. This gives

$$\frac{H}{W}=0,\ \sigma_m \lesseqgtr \sigma_0$$
$$\frac{H}{W}=A\ (\sigma_m-\sigma_0),\ \sigma_m \geqq \sigma_0 \tag{20}$$

Substituting in equation (19)

$$\beta_2 = \tfrac{7}{16}\beta_3\sigma_0$$

$$\beta_4=\beta_5=0$$

we find

$$\frac{H}{W}=\frac{18}{16}\beta_3\left[\sigma_m-\sigma_0-\frac{7}{4}\frac{\sigma_0^2}{\sigma_m}\right]E \tag{21}$$

The third term in the parenthesis can be neglected so that equations (20) and (21) are of the same form. Inserting the value of β_2 and β_3 in equation (17) we obtain

$$f\ (\sigma)=0\ \sigma \lesseqgtr \sigma_0$$
$$f\ (\sigma)=\beta_3\sigma^3\left[1-\frac{9}{16}\frac{\sigma_0}{\sigma}-\frac{7}{16}\left(\frac{\sigma_0}{\sigma}\right)^2\right] \tag{22}$$

where

$$\frac{9}{8}\beta_3=\frac{A}{E}$$

Introducing the value of σ_m from equation (11) equation (20) may be written

$$\frac{H}{W}=\frac{AalL_m}{I}-B \tag{23}$$

$$B=A\sigma_0$$

Thus, when the load amplitude L_m of the alternating flexure cycles is kept contant, the fractional loss of energy is a linear function of the effective length l of the bar.

The deflections d_m and d_0 of the bar at the point of application of the corresponding loads L_m and L_0 are given by the following equations.

$$d_m=\frac{1}{3}\frac{L_m l^3}{IE}=\frac{1}{3}\frac{\sigma_m l^2}{aE}$$
$$d_0=\frac{1}{3}\frac{L_0 l^3}{IE}=\frac{1}{3}\frac{\sigma_0 l^2}{aE} \tag{24}$$

L_0 is the upper limit of the amplitudes of the load cycles for which no appreciable loss of energy is observed. Substituting the value of $\sigma_m-\sigma_0$ obtained from these equations in equation (20), another expression for the energy loss is obtained.

$$\frac{H}{W}=A^3\frac{3Ea}{l^2}\ (d_m-d_0) \tag{25}$$

If D_m and D_0 are the deflections at the end of the bar corresponding to d_m and d_0 (see fig. 3) and if the ratio of the effective length l to the length s_1 of the bar is q, it can be shown that

$$\begin{aligned} l &= q\, s_1 \\ d_m &= \frac{2q}{3-q} D_m \\ d_0 &= \frac{2q}{3-q} D_0 \end{aligned} \tag{26}$$

And by substituting for $d_0 - d_m$ from equation (26) in equation (25) that

$$\frac{H}{W} = \frac{6EaA}{(3q-q^2)s_1^2} (D_m - D_0) \tag{27}$$

If q is equal to unity

$$\frac{H}{W} = \frac{3EaA}{s_1^2} (D_m - D_0) \tag{28}$$

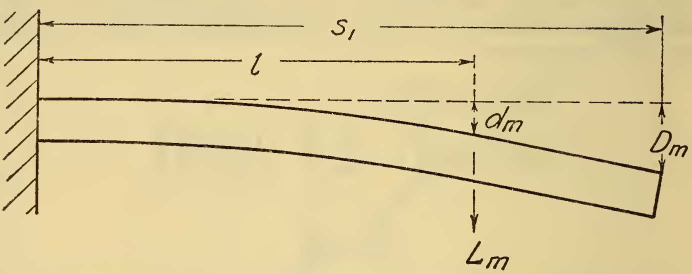

FIGURE 3.—*Diagram of the flexure of a clamped-free bar*

l is the effective length of the bar in flexure under the load L_m.

That is, when the load is applied at the end of the bar the fractional energy loss due to statical hysteresis in the cycles of alternating flexure varies linearly with the maximum deflection D_m. D_0 is the upper limit of the deflection amplitudes of the cycle at which practically no hysteresis is observed. The following relation between σ_0 and D_0, when q is unity, is obtained from equation (24) after obvious substitutions are made

$$\sigma_0 = \frac{3aE}{s_1^2} D_0 \tag{29}$$

2. DURING ALTERNATING FLEXURE PRODUCED BY LATERAL VIBRATION

The expressions for the energy loss due to statical hysteresis in cycles of alternating statical flexure derived above are readily applicable to the loss of energy per cycle in the lateral vibrations of the bar, provided the vibrations are of the fundamental mode. The transfer of expressions from the one case to the other is made possible by the fact that the configuration of a vibrating bar can be approximated with sufficient closeness by a statical deformation of the same bar.[14] It is sufficient for this purpose to deflect the bar by applying

[14] Rayleigh, The Theory of Sound, vol. 1, 2d ed., p. 284.

a single load at a distance from the free end equal to one-fourth of the length of the bar; that is, if the length of the bar is s_1, the effective length l to be chosen is

$$l = \tfrac{3}{4} s_1$$

Let α be the amplitude of the excursions of the end of the bar, W be the energy of deformation of the bar at this amplitude, and H be the loss of energy due to statical hysteresis. The fractional loss of energy per cycle due to statical hysteresis, $\frac{H}{W}$, can then be evaluated by substituting the following quantities into equation (27)

$$q = \tfrac{3}{4}$$
$$\alpha = D_m$$
$$\alpha_0 = D_o$$

This gives

$$\frac{H}{W} = \frac{32}{9} \frac{EaA}{s_1^2} [\alpha - \alpha_0] \tag{30}$$

Here α_0 represents the upper limit of the amplitudes of the vibrations which gives practically no damping. But the fractional loss of energy $\frac{H}{W}$ is

$$\frac{H}{W} = \frac{2d\alpha}{\alpha dn}$$

where $\frac{d\alpha}{dn}$ is the decrement of the amplitude per vibration. Hence

$$\frac{2\,d\alpha}{\alpha\,dn} = \frac{32}{9} \frac{Ea}{s_1^2} (\alpha - \alpha_0)\,A \tag{31}$$

Thus, in the lateral vibration of the bar the fractional loss of energy per cycle, due to statical hysteresis, is a linear function of the amplitude of vibration. The relation of α_0 to σ_0 is derived from equations (24) and (26).

$$\sigma_0 = \frac{32}{9}\, a \,\frac{E}{s_1^2}\, \alpha_0 \tag{32}$$

III. APPLICATION OF THE THEORY TO A U BAR

In a previous paper,[15] it is shown that the states of deformation of an elongated U bar in vibration and of a vibrating straight clamped-free bar are practically alike. Consider, for example, the frequency of vibration. If the ratio of the median length of the yoke to the total median length of the U bar equals one-tenth, the frequency of vibration of the U bar will be only 0.5 per cent higher than that of a straight clamped-free bar with a length the same as the median length of each arm of the U bar. Thus, all the equations for the statical hysteresis of a clamped-free bar can be applied to an elongated U bar, with only a small error caused by the yoke configuration, provided that the axial or median length of the straight clamped-free bar, measured from the fixed end, is the same as the median length of the arms of the U bar, measured from the mid-point of the yoke. (See fig. 4.)

[15] See footnote 3, p. 635.

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IV. APPARATUS AND EXPERIMENTAL PROCEDURE

1. FOR MEASUREMENTS DURING A LOAD CYCLE

The essential features of the apparatus used to measure the statical hysteresis in the alternating flexure of U bars produced by a load cycle are shown in Figure 5. The fork of the U bar *A* is gripped by the two clamps *B* and *C*. A transmitting rod *R* is attached to clamp *B* in a direction normal to the axis of the U bar. Forces *L* act normally at the ends of the transmitting rod *R* and are of two kinds; first, the weights on the panel *P* pull the rod downwards and second, the buoyant force of mercury in the basins acting on the floats attached to the end of rod *R* push it upward. Actually four mercury basins were used although only two are shown in Figure 5. The value of *L* is varied by varying the load on the panel *P*.

The lower clamp *C* is free to rotate about axis *ff* in the cradle *D*, which is attached firmly to a heavy base plate *E*. The arms *F* extend

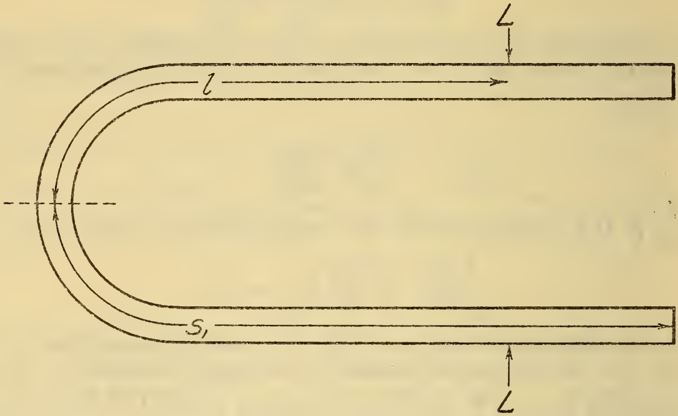

FIGURE 4.—*Diagram of the U bar*

out from the lower portion of clamp *C* and is used to control the direction of the axis of the U bar in space. Before each measurement of the deflection of the U bar is made, *F* is adjusted by the screw support until the principal axis of the U bar is horizontal. The sensitive spirit level *h* serves to determine this position. The micrometer *M* measures the displacement of the upper prong relative to the lower prong, at the points where the load *L* is applied. The contact between the micrometer and the upper prong is determined electrically. In most of the experiments the error caused by uncertainty of the contact was not greater than 0.0002 mm.

2. DYNAMICAL MEASUREMENT DURING LATERAL VIBRATIONS

For the study of the damping of the vibrations of the U bar, photographic records of the amplitudes of the end excursions were obtained by permitting a beam of light to traverse first a slit in a vane mounted on one of the prongs, then a spherical lens, and finally a cylindrical lens before reaching the photographic paper. This paper was mounted on a rotating wheel whose axis was parallel to the direction of the lateral vibrations of the prongs. Since the beam

of light emerging from the slit was in the form of a ribbon, the cylindrical lens was used to focus the line section of the ribbon to a sharp point. The magnification employed was in the neighborhood of 10.

For these experiments the U bar was clamped at the yoke, in the manner described later in the section on experimental results.

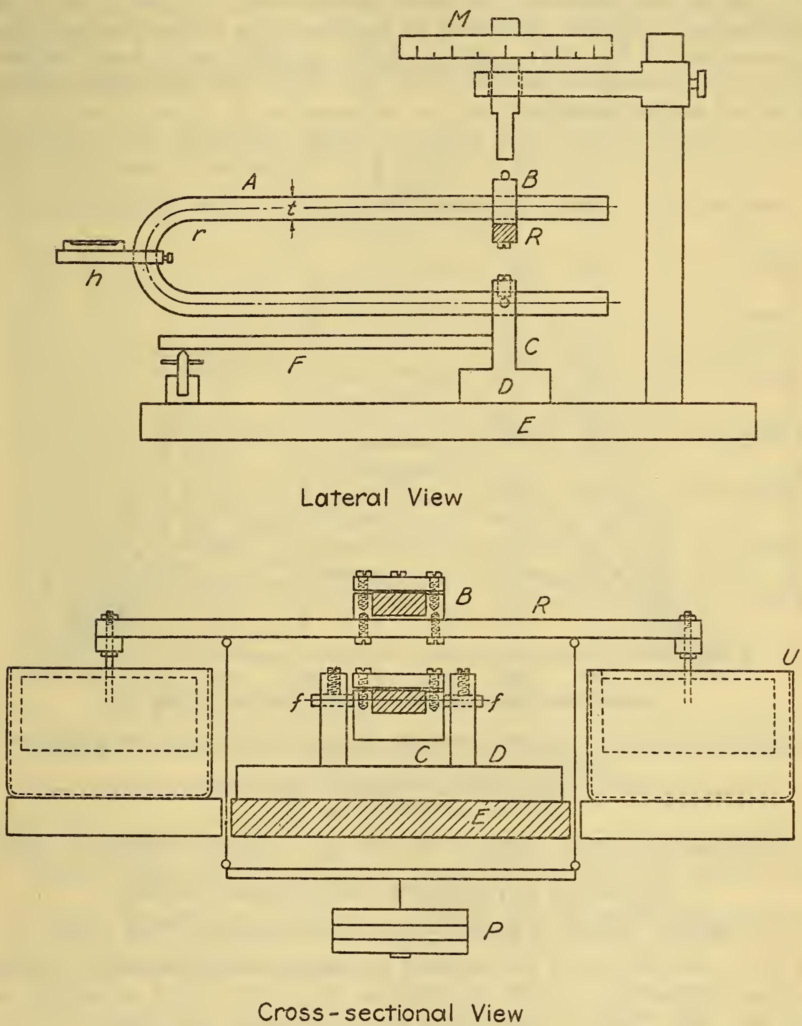

FIGURE 5.—*Diagram of the apparatus used in connection with the statical measurements*

L shows the point of application of the load; *l* is the effective length and s_1, the length.

3. DESCRIPTION OF THE U BAR

The U bar used consisted of a circular yoke and two parallel prongs as shown in Figures 4 and 5. The radius of the yoke was 1.69 cm. The total median length of the bar was 64.30 cm; that is, each arm from the center of the yoke was 32.15 cm in median length. The average thickness $2a$ of the two prongs was 0.829 and

0.825 cm, respectively. The variation in thickness from these mean values was less than ±0.003 cm. The mean width of the bar was 1.393 cm with variations not exceeding ±0.002 cm. In the symbols of the paper $s_1=32.15$ cm, $a=0.414$ cm, $b=1.393$ cm. The bar was made of Armco iron.

In the initital part of the investigation a cylindrical stem of a diameter equal to the thickness of the bar was joined to the yoke at the center of the convex side. After carrying out a few observations on the damping of this fork, the stem was detached, giving the U bar above described.

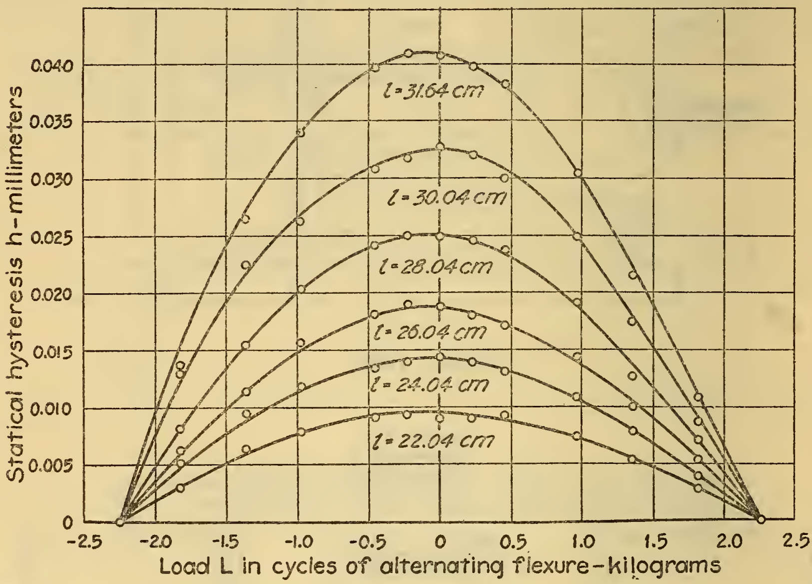

FIGURE 6.—*Values of statical hysteresis observed during the alternate flexure cycles of an armco iron* U *bar*

During these cycles the load amplitude L_m had the constant value of 2.27 kg; the effective length l was varied.

V. EXPERIMENTAL RESULTS ON AN ARMCO IRON U BAR

1. STATICAL ALTERNATE CYCLES OF FLEXURE

(a) ENERGY LOSS AS A FUNCTION OF THE LENGTH OF THE U BAR

Let us consider first the results of statical experiments in which L_m, the amplitude of the alternating load, was kept constant at 2.27 kg. The effective length l of the U-bar arms was changed from 31.64 cm to 30.04, 28.04, 26.04, 24.04, and 22.04 cm in turn.

In performing the cycle we started with $L=0$, increasing to $L=+L_m$, then decreasing to $L=-L_m$ and finally increasing to $L=0$. Each alternating load cycle was repeated three times. The elastic lag or "the width of the hysteresis loop" was determined as the difference of the deflections at any load L for increasing and decreasing values. It was found that the after effect, or the difference in deflection for the initial and final zero loads in these various cycles was practically zero. It was therefore considered safe to assume that the elastic lag was due almost exclusively to statical hysteresis and

that hysteresis due to elastic afterworking was practically absent. The results are given in Table 1 where the statical hysteresis h is given corresponding to each load L of the load cycle.

The data in Table 1 are plotted in Figure 6, where the circles are observed values. The areas under these curves give the loss of energy H due to statical hysteresis in the two arms of the U bar for the various load cycles. The values of H in ergs are given in Table 2.

We now need the values of W for computing $\frac{H}{W}$, where W is the maximum energy of deformation of the whole bar when L has the value L_m. It is known that W equals the product $d_m L_m$, d_m being the deflection of the U bar at the point of application of the load L_m. The calculated values of W and $\frac{H}{W}$ are also given in Table 2.

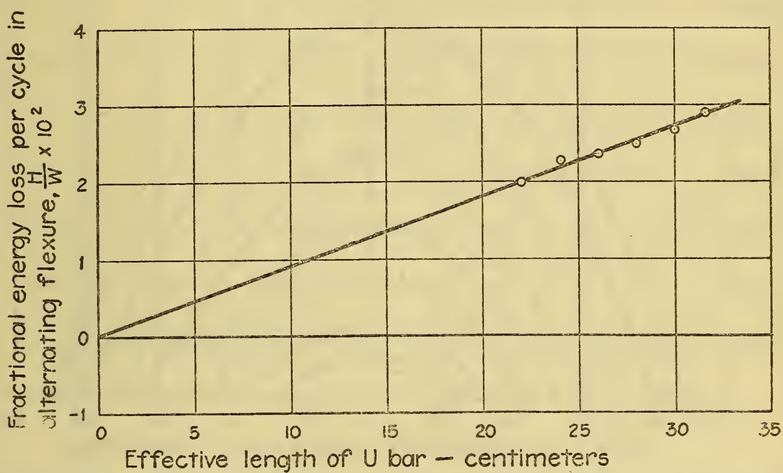

FIGURE 7.—*Variation during alternating flexure cycles of the fractional energy loss* $\frac{H}{W}$ *with the effective length* l *of the* U *bar*

The load amplitude L_m was 2.27 kg.

These values of $\frac{H}{W}$ are plotted against l, the effective length of the arms of the bar, in Figure 7. Within the experimental error the data can be represented by a straight line. The method of least squares gives for the equation of the line

$$\frac{H}{W} \times 10^{2} = 0.092l - B \qquad (33)$$

$$B = 0.000$$

Owing to the great extrapolation necessary there is considerable uncertainty in the value of B, and this experiment may be said to leave its value indeterminate. Equation (33) agrees with theoretically derived equation (23) and, therefore, the results of the experiment are not inconsistent with the hysteresis law as given by equation (22).

(b) ENERGY LOSS AS A FUNCTION OF THE MAXIMUM DEFLECTION

Let us consider next the results of experiments in which the effective length l was kept constant at the value 31.64 cm, practically the length of the arms of the U bar. The load amplitude, L_m, of the alternating flexure cycles was varied from 2.268 kg to 2.041, 1.814, 1.588, 1.361, 1.134, 0.907, and 0.690 kg, respectively. For these conditions the statical hysteresis h was determined. These values are given in Table 3, for each load L of the load cycle. The data of this table are presented in the plots of Figure 8 in which the circles repre-

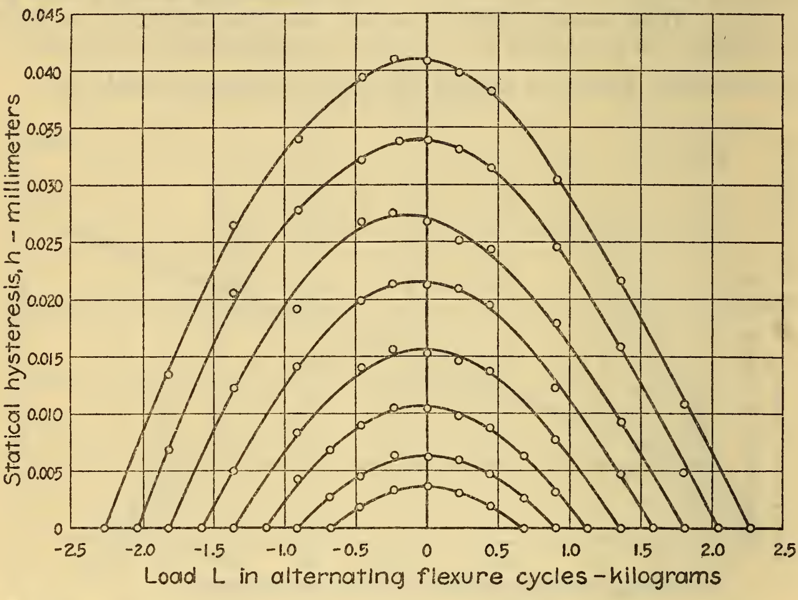

FIGURE 8.—*Experimental values of the statical hysteresis in alternate flexure cycles of an Armco iron U bar 31.64 cm in effective length for various load amplitudes L_m*

sent observed values. The values of H in ergs, as calculated from the areas under these curves are entered in Table 4 as well as the values of $d_m(=D_m,$ because the load was applied at the end of the bar$)L_m$, and the computed values of W and $\frac{H}{W}$.

The values of $\frac{H}{W}$ given in Table 4 are plotted against D_m in Figure 9. Within the experimental error the data can be represented by a straight line which meets the axis at a deflection $D_m=D_o$. The method of least squares gives for the equation of the line

$$\frac{H}{W}\times 10^2=1.70\ (D_m-D_o) \tag{34}$$

$$D_o=0.04\ mm$$

This is in agreement with the theoretical result, equation (23), and therefore the results of the experiment are consistent with the hysteresis law as given by the equation (22).

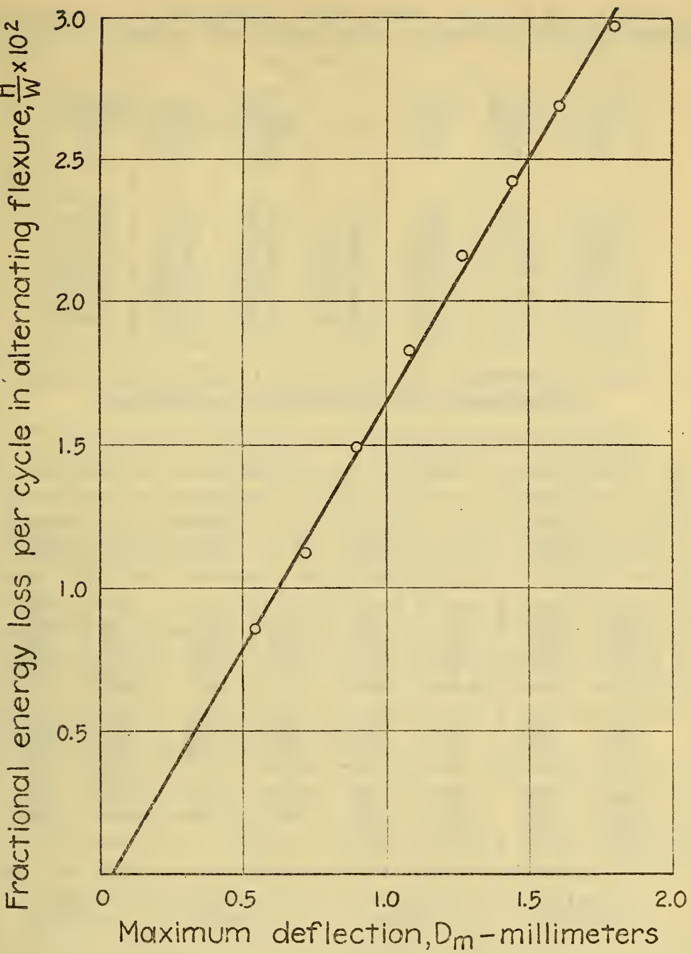

FIGURE 9.—*Variation of the fractional loss of energy* $\frac{H}{W}$ *during alternating flexure cycles of an Armco iron* U *bar 31.64 cm in effective length as a function of the maximum deflection* D_m

TABLE 1.—*Statical hysteresis in an Armco iron* U *bar of variable length in alternating flexure cycles with constant load amplitude*

L_m, kg / l, cm	2.27 / 31.64	2.27 / 30.04	2.27 / 28.04	2.26 / 26.04	2.27 / 24.04	2.27 / 22.04
Load L	Statical hysteresis h					
kg	*mm*	*mm*	*mm*	*mm*	*mm*	*mm*
−2.268	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−1.814	.0136	.0129	.0082	.0062	.0052	.0030
−1.361	.0265	.0225	.0154	.0115	.0095	.0064
−.907	.0340	.0263	.0204	.0157	.0119	.0078
−.453	.0396	.0309	.0242	.0181	.0135	.0091
−.227	.0410	.0318	.0250	.0190	.0140	.0095
.000	.0408	.0328	.0249	.0188	.0146	.0090
.227	.0398	.0321	.0246	.0181	.0140	.0091
.453	.0382	.0300	.0237	.0172	.0132	.0093
.907	.0304	.0249	.0192	.0145	.0109	.0074
1.361	.0216	.0174	.0126	.0101	.0078	.0053
1.814	.0109	.0087	.0071	.0053	.0039	.0028
2.268	.0000	.0000	.0000	.0000	.0000	.0000

TABLE 2.—*Fractional energy loss due to statical hysteresis in an Armco iron* U *bar of variable length in alternating flexure cycles with constant load amplitude*

Effective length	Load amplitude L_m	Deflection d_m	$L_m d_m$	Energy of the deformation (in ergs) W	Hysteresis loss (in ergs) H	Fractional energy loss $\frac{H}{W}$
cm	*kg*	*mm*		$\times 10^2$		$\times 10^{-2}$
31.64	2.268	1.804	4.090	4,010	11,900	2.968
30.04	----------	1.545	3.502	3,440	9,310	2.708
28.04	----------	1.257	2.850	2,791	7,130	2.555
26.04	----------	1.005	2.210	2,232	5,352	2.398
24.04	----------	.782	1.773	1,747	4,020	2.300
22.04	----------	.611	1.385	1.356	2,698	1.988

TABLE 3.—*Statical hysteresis in an Armco iron* U *bar of constant length in alternating flexure cycles with variable load amplitude*

L_m=2.268 kg l=31.54 cm		L_m=2.041 kg l=31.64 cm		L_m=1.814 kg l=31.64 cm		L_m=1.588 kg l=31.54 cm	
Load L	Statical hysteresis h	Load L	Statical hysteresis h	Load L	Statical hysteresis h	Load L	Statical hysteresis h
kg	*mm*	*kg*	*mm*	*kg*	*mm*	*kg*	*mm*
−2.268	0.0000	−2.041	0.0000	−1.814	0.0002	−1.588	0.000
−1.814	.0136	−1.814	.0068	−1.361	.0123	−1.361	.0050
−1.361	.0265	−1.361	.0206	−.907	.0192	−.907	.0131
−.907	.0340	−.907	.0278	−.454	.0268	−.454	.0198
−.453	.0396	−.454	.0322	−.227	.0275	−.227	.0213
−.227	.0410	−.227	.0337	.000	.0268	.000	.0213
.000	.0408	.000	.0339	.227	.0252	.227	.0207
.227	.0398	.227	.0331	.454	.0245	.454	.0195
.453	.0382	.454	.0315	.907	.0178	.907	.0123
.907	.0304	.907	.0246	1.361	.0093	1.361	.0047
1.361	.0216	1.361	.0159	1.814	.0000	1.588	.0000
1.814	.0109	1.814	.0049				
2.268	.0000	2.041	.0000				

L_m=1.361 kg l=31.64 cm		L_m=1.134 kg l=31.64 cm		L_m=0.907 kg l=31.64 cm		L_m=0.680 kg l=31.64 cm	
Load L	Statical hysteresis h	Load L	Statical hysteresis h	Load L	Statical hysteresis h	Load L	Statical hysteresis h
kg	*mm*	*kg*	*mm*	*kg*	*mm*	*kg*	*mm*
−1.361	0.0000	−1.134	0.0000	−0.907	0.0000	−0.680	0.0000
−.907	.0083	−.907	.0043	−.680	.0027	−.454	.0019
−.454	.0141	−.680	.0068	−.454	.0046	−.227	.0034
−.227	.0156	−.454	.0090	−.227	.0063	.000	.0036
.000	.0152	.227	.0105	.000	.0062	.227	.0032
+.227	.0146	.000	.0105	.227	.0059	.454	.0020
.454	.0138	.227	.0099	.454	.0048	.680	.0000
.907	.0078	.454	.0088	.680	.0026		
1.361	.0000	.680	.0063	.407	.0000		
		.907	.0032				
		1.134	.0000				

TABLE 4.—*Fractional energy loss due to statical hysteresis in an Armco iron* U *bar of constant length in alternating flexure cycles with variable load amplitude*

Effective length	Load amplitude L_m	Deflection $d_m(=D_m)$	$L_m d_m$	Energy of deformation (in ergs) W	Hysteresis loss (in ergs) H	Fractional energy loss $\frac{H}{W}$
cm	*kg*	*mm*		$\times 10^3$		$\times 10^{-2}$
31.64	0.680	0.542	0.368	361	303	0.838
----------	.907	.722	.654	641	721	1.122
----------	1.134	.903	1.023	1,022	1,526	1.494
----------	1.361	1.083	1.475	1,445	2,654	1.836
----------	1.588	1.262	2.002	1,961	4,221	2.150
----------	1.814	1.443	2.617	2,565	6,187	2.412
----------	2.041	1.624	3.315	3,248	8,720	2.685
----------	2.268	1.804	4.090	4,010	11,900	2.968

2. ENERGY LOSS DURING VIBRATION

(a) EFFECT OF METHOD OF SUPPORTING THE U BAR

In the experiments on the damping of the vibrations of the U bar it was necessary to consider the effect of the method of supporting the bar. The various methods of support were as follows: First, the stem initially forming part of the yoke was firmly clamped to a heavy rigid base. Second, the stem was inserted into a rubber bed, the bed being surrounded by a metal cylinder 5 cm in diameter, which in turn was firmly attached to a heavy base. Third, the U bar was hung vertically downward from the stem being supported by a strand of ordinary thread. Fourth, the stem was detached and the U bar was hung vertically downward by means of a narrow wooden vise gripping the yoke at its mid portion.

With the support as in the third method, damping measurements were made with the unloaded bar, and then with a 4 g weight attached successively to the extremities of the prongs. With the support as in the fourth method, damping measurements were made both before and after the statical tests discussed in the preceding section, in order to determine whether these tests affected the damping.

The double amplitudes of vibration, 2α, measured from the photographic records were plotted against the number of vibrations n. The smooth curves drawn through these plotted values were taken to represent the damping curve of the vibrations. The double amplitudes of vibration as read from these curves are given in Table 5 for the four methods of supporting the U bar. The data in columns 3a and 3b in the table headed 3, refer to the third method of support, column 3 being for the case in which the prongs were unloaded, and columns 3a and 3b for that in which the prongs were loaded. The data in columns 4 and 4a refer to the fourth method of support, the first before, and the latter after, the statical tests.

In order to obtain the fractional energy loss per cycle for any particular cycle which is

$$\frac{2}{a}\frac{d\alpha}{dn}$$

the slopes $2\,\frac{d\alpha}{dn}$ of the damping curves were determined graphically and were plotted against α. A smooth curve was drawn in each case, thus obtaining the curve of decrements. The curve thus

obtained from the data for the fourth method of supporting the U bar is given in Figure 10. Division of the ordinate by the abscissa of the decrement curve gives, for any desired value of the amplitude a, the fractional loss in energy. The values thus obtained are presented in Table 6.

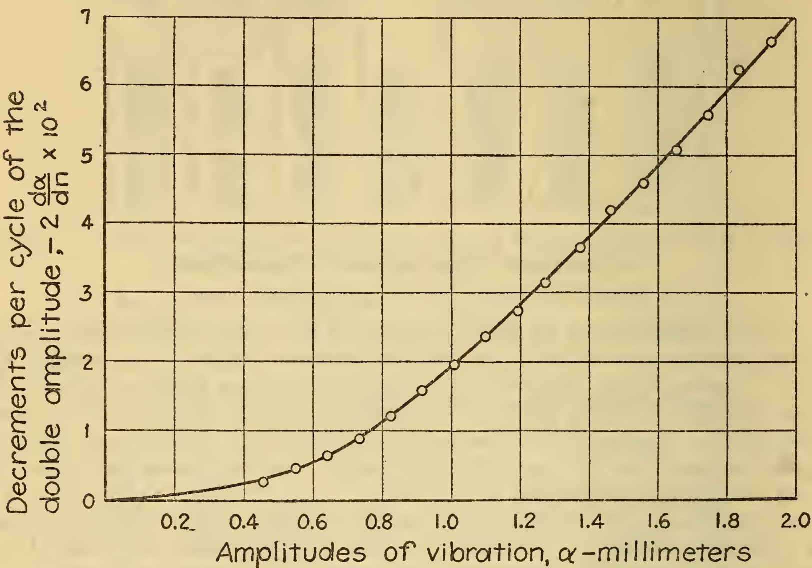

FIGURE 10.—*A typical curve showing the decrement per complete cycle at various amplitudes of vibration of the freely vibrating* U *bar*

TABLE 5.—*Double amplitudes of vibration of the Armco iron* U *bar for various methods of support*

Vibration No.	Method of support						
	1	2	3	3a	3b	4	4a
	Double amplitude of the end excursions 2a						
n	*mm*	*mm*	*mm*	*mm*	*mm*	*mm*	*mm*
0	3.200	4.520	4.480	4.065	4.305	5.310	3.995
10	2.760	3.765	3.760	3.485	3.635	4.402	3.397
20	2.432	3.250	3.265	3.033	3.132	3.712	2.950
30	2.175	2.848	2.875	2.685	2.742	3.202	2.610
40	1.972	2.543	2.570	2.400	2.450	2.809	2.345
50	1.808	2.295	2.335	2.182	2.210	2.480	2.125
60	1.670	2.090	2.140	2.003	2.018	2.248	1.940
70	1.550	1.922	1.972	1.855	1.852	2.045	1.790
80	1.452	1.782	1.832	1.730	1.718	1.880	1.665
90	1.368	1.661	1.720	1.618	1.600	1.748	1.562
100	1.295	1.560	1.615	1.520	1.520	1.632	1.462
120	1.172	1.410	1.437	1.375	1.375	1.453	1.327
140	1.072	1.293	1.310	1.255	1.252	1.320	1.222
160	.995	1.196	1.218	1.160	1.148	1.210	1.134
180	.930	1.113	1.138	1.085	1.068	1.118	1.060
200	.876	1.050	1.070	1.022	1.002	1.038	.992
220	.828	.983	1.008	.960	.948	.975	.928
240	.785	.923	.958	.905	.898	.930	.873
260	.747	.868	.913	.860	.853		.825
280		.819	.872	.823	.813		.785
300		.778	.840	.788	.775		.752
320		.742	.808	.760	.738		.722

TABLE 6.—*Fractional loss of energy per cycle of lateral vibration of the Armco iron U bar*

Amplitude α	Method of support							
	1	2	3	3a	3b	4	4a	Average
	Fractional loss in energy $\frac{2}{\alpha}\frac{d\alpha}{dn}\times 10^2$							
mm								
2.0	4.20	4.02	3.72	3.68	3.68	3.60	3.76	3.84
1.9	4.00	3.72	3.52	3.50	3.54	3.46	3.60	3.62
1.8	3.78	3.50	3.30	3.32	3.38	3.32	3.42	3.44
1.7	3.54	3.28	3.08	3.14	3.22	3.16	3.28	3.24
1.6	3.30	3.06	2.86	2.94	3.04	3.00	3.08	3.04
1.5	3.04	2.82	2.66	2.74	2.88	2.84	2.88	2.84
1.4	2.80	2.62	2.44	2.56	2.68	2.68	2.68	2.64
1.3	2.56	2.32	2.26	2.36	2.48	2.52	2.50	2.42
1.2	2.32	2.18	2.04	2.16	2.30	2.32	2.30	2.22
1.1	2.06	1.96	1.86	1.94	2.08	2.14	2.08	2.02
1.0	1.84	1.76	1.64	1.76	1.76	1.80	1.86	1.78
.9	1.62	1.58	1.46	1.56	1.62	1.68	1.64	1.60
.8	1.40	1.34	1.24	1.34	1.40	1.42	1.40	1.36
.7	1.18	1.12	1.06	1.12	1.14	1.12	1.14	1.12
.6	.94	.94	.82	.86	.88	.86	.88	.88
.5	.70	.74	.62	.66	.68	.68	.64	.64
.4	.54	.54	.42	.42	.44	.44	.42	.44
.3	.30	.36	.30	.26	.30	.30	.30	.30

When the data in the various columns in Table 6 for a given amplitude α are examined one finds that individual values differ as much as 10 per cent from the average value. Since the data in column 3 were obtained with a support possessing practically no elastic coupling with the fork, it is expected that the values of the fractional loss of energy given in column 3 will be smaller than those in the other columns. Actually this is the case, but the variation is less than 4 per cent from the average value. On the other hand, the data may have errors as high as 5 per cent due to slight shifting of the vibrating U bar from experiment to experiment causing variations in the value of the magnification of the optical system from the value used in reducing the data. Since the experimental error is of about the same order of magnitude as the observed average effect of the various methods of supporting the bar, it appears reasonable to ignore the effect of the various supports on the energy loss. The average value of the energy loss is therefore used in making the final calculations.

(b) ENERGY LOSS PER CYCLE

Values of the average fractional loss in energy given in Table 6 are plotted against the amplitude α in Figure 11. For amplitudes equal to or greater than 0.9 mm, these data fall on a straight line which meets the axis at the amplitude $\alpha = \alpha_0$. The points at the smaller amplitudes are neglected in determining this line because (*a*) the experimental errors are relatively very large at these amplitudes and (*b*) primarily because the theory which has been developed applies only to large values of the stress amplitude (equation (19)). The equation of this line, determined by least squares, is as follows:

$$\frac{2}{\alpha}\frac{d\alpha}{dn}\times 10^2 = 2.02\ (\alpha - \alpha_0) \tag{35}$$

$$d_0 = 0.10\ mm$$

This is in agreement with equation (31) and therefore the results of the experiment are consistent with the hysteresis law of the equation (22).

VI. COMPARISON OF THE EXPERIMENTAL RESULTS

Thus far we have seen that the results of the three typical tests considered are qualitatively in good agreement with the theory.

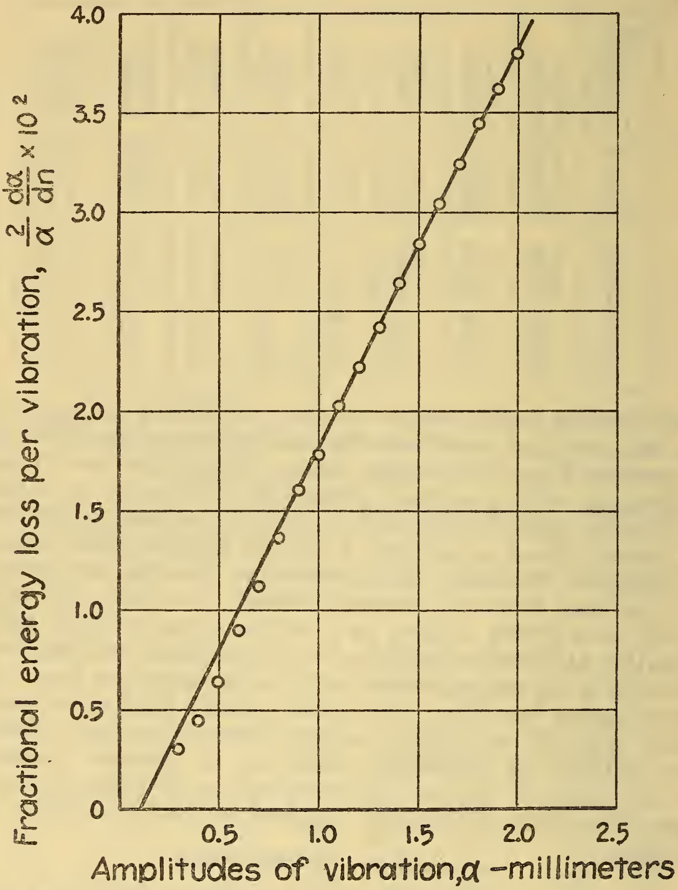

FIGURE 11.—*Variation of the fractional energy loss per complete vibration with amplitude of vibration during the lateral vibration of an Armco iron U bar*

That is, tests of each type are consistent with equations (22). It now remains to see how well the various results agree quantitatively.

It is seen that a comparison of equation (23) and the experimental results expressed in equation (33) gives the first of the following relations. A similar comparison of the theoretically derived equations (28) and (31) with the corresponding experimental results in equations (34) and (35) gives the last two of the relations below.

$$\frac{aL_mA}{I} = 0.92 \times 10^{-3}, \ (L_m = 2.27 \text{ kg})$$

$$\frac{3\ EaA}{s_1^2}=1.70\times 10^{-1} \text{ per cm}$$

$$\frac{32\ EaA}{9\ s_1^2}=2.02\times 10^{-1} \text{ per cm}$$

The first two relations are based on the two series of results obtained statically and the third on the results obtained from the vibrating U bar. Introducing the numerical values of constants

$$s_1=31.64 \text{ cm}$$

$$E=2.03\times 10^6 \text{ kg per cm}^2$$

$$a=0.413 \text{ cm}$$

$$b=1.393 \text{ cm}$$

$$I=6.25\times 10^{-2} \text{ cm}^4$$

it is found that the corresponding determinations of A are

$$\begin{aligned} A&=0.66\times 10^{-4} \text{ cm}^2 \text{ per kg}\\ A&=0.68\times 10^{-4} \text{ cm}^2 \text{ per kg}\\ A&=0.71\times 10^{-4} \text{ cm}^2 \text{ per kg}\end{aligned} \tag{36}$$

Next for the determination of the small stress σ_0 below which practically no energy loss occurs, we have the three independent relations

$$\sigma_0=\frac{B}{A};\ B=0.00\times 10B^{-2}$$

$$\sigma_0=\frac{3\ aE}{s_1^2}\ D_0;\ D_0=0.4\times 10^{-2} \text{ cm}$$

$$\sigma_0=\frac{32}{9}\ a\ \frac{E}{s_1^2}\alpha_0\ \ \alpha_0=1.0\times 10^{-2} \text{ cm}$$

These have been, respectively, obtained from the theoretical results in equations (23), (29), and (32) and the experimental results given in equations (33), (34), and (35). Calculation gives, using the constants given in the previous paragraph,

$$\begin{aligned} \sigma_0&=00 \text{ kg per cm}^2\\ \sigma_0&=10 \text{ kg per cm}^2\\ \sigma_0&=27 \text{ kg per cm}^2\end{aligned} \tag{37}$$

Hence, we see that the results of the three typical experiments are qualitatively in agreement, since they all give practically the same value for A. Only in the case of σ_0 is the discrepancy between the various determinations considerable. This is to be expected, since σ_0 was obtained by extrapolations of considerable uncertainty from test to test. As a probable value we suggest $\sigma_0=20$ kg per cm^2, ignoring the value of $\sigma_0=0$ since the extrapolation by which it was obtained has by far the greatest uncertainty.

We, therefore, conclude that the agreement between these three typical tests shows that the theory given furnishes an adequate basis for determining the statical hysteresis of materials of this type by measurement of the decrement of freely vibrating bars.

VII. LAW OF STATICAL HYSTERESIS IN ARMCO IRON

On the basis of the theory developed and the experimental results obtained the law of statical-hysteresis in Armco iron for stresses at least up to 600 kg per cm² appears to be of the form

$$f(\sigma)=0 \quad \sigma \lesseqqgtr \sigma_0$$

$$f(\sigma)=\beta_3\sigma^3\left[1-\frac{9}{16}\frac{\sigma_0}{\sigma}-\frac{7}{16}\left(\frac{\sigma_0}{\sigma}\right)^2\right] \tag{38}$$

The value of σ_0 was found to be 20 kg per cm². From equation (22)

$$\beta_3=\frac{8}{9}\frac{A}{E}$$

With the mean value of A 0.68×10^{-4} cm² per kg, the numerical value of β_3 now becomes

$$\beta_3=2.91\times10^{-11}\text{ cm}^4\text{ per kg}^2$$

or

$$\beta_3=2.91\times10^{-5}\text{ ergs cm}^3\text{ per kg}^3 \tag{39}$$

The results on hysteresis loss in Armco iron as given in previous papers[16] were obtained with a bar cut from the same sample as the present one under discussion. In these papers it was assumed that

$$f(\sigma)=\frac{1}{3}\beta\sigma_r^3 \tag{40}$$

where

$$\beta=1.15\times10^{-5}\text{ ergs cm}^3\text{ per kg}^3$$

and σ_r represents the range of stresses in the cycles, the cycles being alternate or otherwise. Putting

$$\sigma_r=2\sigma$$

it follows that

$$\beta_3=\frac{8}{3}\beta$$

or in the symbols of this report (40) becomes

$$f(\sigma)=\beta_3\sigma^3 \tag{41}$$

where

$$\beta_3=3.07\times10^{-5}\text{ ergs cm}^3\text{ per kg}^3$$

Comparing the expressions (38, and (41), the values of β_3 are seen to be substantially equal, but the threshold stress σ_0 does not appear in equation (41). This absence of σ_0 is explained by the fact that the method of calculations adopted in the previous papers and the lack of sufficient data obscured the presence of a threshold stress in the energy loss due to statical hysteresis.

WASHINGTON, March 19, 1932.

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[16] See footnotes 1 and 2, p. 635.